DEMOIVRE – COMPLEX NUMBERS – ROOTS REVIEW

1. Consider the complex number $z = \cos\theta + \text{i}\sin\theta$.

(a) Using De Moivre's theorem show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta.$$

(2)

(b) By expanding $\left(z + \frac{1}{z}\right)^4$ show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

(4)

(c) Let $g(a) = \int_0^a \cos^4\theta \, d\theta$.

(i) Find $g(a)$.

(ii) Solve $g(a) = 1$

(5)
(Total 11 marks)

2. Consider the complex number $z = \dfrac{\left(\cos\frac{\pi}{4} - \text{i}\sin\frac{\pi}{4}\right)^2\left(\cos\frac{\pi}{3} + \text{i}\sin\frac{\pi}{3}\right)^3}{\left(\cos\frac{\pi}{24} - \text{i}\sin\frac{\pi}{24}\right)^4}$.

(a) (i) Find the modulus of z.

(ii) Find the argument of z, giving your answer in radians.

(4)

(b) Using De Moivre's theorem, show that z is a cube root of one, *i.e.* $z = \sqrt[3]{1}$.

(2)

(c) Simplify $(1 + 2z)(2 + z^2)$, expressing your answer in the form $a + b\text{i}$, where a and b are **exact** real numbers.

(5)
(Total 11 marks)

3. Let the complex number z be given by

$$z = 1 + \frac{\text{i}}{\text{i} - \sqrt{3}}.$$

Express z in the form $a + b\text{i}$, giving the **exact** values of the real constants a, b.

(Total 6 marks)

4. Given that $(a + i)(2 - bi) = 7 - i$, find the value of a and of b, where $a, b \in \mathbb{Z}$.

(Total 6 marks)

5. Given that $|z| = 2\sqrt{5}$, find the complex number z that satisfies the equation

$$\frac{25}{z} - \frac{15}{z^*} = 1 - 8i.$$

(Total 6 marks)

6. Given that $z = (b + i)^2$, where b is real and positive, find the **exact** value of b when $\arg z = 60°$.

(Total 6 marks)

7. A complex number z is such that $|z| = |z - 3i|$.

(a) Show that the imaginary part of z is $\frac{3}{2}$.

(2)

(b) Let z_1 and z_2 be the two possible values of z, such that $|z| = 3$.

(i) Sketch a diagram to show the points which represent z_1 and z_2 in the complex plane, where z_1 is in the first quadrant.

(ii) Show that $\arg z_1 = \frac{\pi}{6}$.

(iii) Find $\arg z_2$.

(4)

(c) Given that $\arg\left(\frac{z_1^k z_2}{2i}\right) = \pi$, find a value of k.

(4)

(Total 10 marks)

8. $(z + 2i)$ is a factor of $2z^3 - 3z^2 + 8z - 12$. Find the other two factors.

(Total 3 marks)

9. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and $c \in \mathbb{R}$. Two of the roots of $P(z) = 0$ are -2 and $(-3 + 2i)$. Find the value of a, of b and of c.

(Total 6 marks)